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TESTING FOR REGIME SWITCHING: A COMMENT

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An autoregressive model with Markov regime-switching is analyzed that reflects on the properties of the quasi-likelihood ratio test developed by Cho and White (2007). For such a model, we show that consistency of the quasi-maximum likelihood estimator for the population parameter values, on which consistency of the test is based, does not hold. We describe a condition that ensures consistency of the estimator and discuss the consistency of the test in the absence of consistency of the estimator.

KEYWORDS: Consistent, Markov regime switching, quasi-maximum likelihood.

1. INTRODUCTION

IN CHO AND WHITE (2007), "Testing for Regime Switching," the authors studied the asymptotic behavior of a statistic that tests the null hypothesis of one regime against the alternative of Markov switching between two regimes. A key insight is that a consistent test can be based on a quasi-likelihood that ignores the Markov structure of regime switching and treats the state variables that indicate regimes as a sequence of independent and identically distributed random variables. Consistency of the test follows from consistency of the quasi-maximum likelihood estimator (QMLE) under the alternative, which appears as Theorem 1(b) in Cho and White. Consistency of the QMLE requires that the expected quasi-log-likelihood attain a global maximum at the population parameter values. We show that this requirement does not hold for the autoregressive process analyzed in Cho and White. Thus, for models of regime switching in which the conditional mean contains autoregressive components, consistency of the test proposed by Cho and White has not been established.

For the observable random variables $\{X_t \in \mathbb{R}^d\}_{t=1}^n$, $d \in \mathbb{N}$, the Markov regime-switching autoregressive process analyzed by Cho and White (Section 3, p. 1697) is

$$X_t = \theta_* \cdot 1_{\{S_t=1\}} - \theta_* \cdot 1_{\{S_t=2\}} + .5X_{t-1} + u_t, \tag{1}$$

where $u_t \sim$ i.i.d. $N(0, 1)$ and $S_t \in \{1, 2\}$ is the sequence of unobserved state variables that indicate regimes. The insight of Cho and White is to replace the conditional state probability $\mathbb{P}(S_t = 2|\sigma(X^{t-1}))$, where $\sigma(X^{t-1})$ is the smallest σ-algebra generated by $X^{t-1} := (X'_{t-1}, \dots, X'_1)$, with the unconditional probability from the stationary distribution of the state variables, $\pi = \mathbb{E}[\mathbb{P}(S_t = 2|\sigma(X^{t-1}))]$. The resulting quasi-likelihood simplifies the model, although serial correlation in the state variables is ignored. Ignoring this serial correlation

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can lead to inconsistency if the conditional state probabilities depend upon the regressors that enter the state-specific conditional densities.

The quasi-log-likelihood employed by Cho and White for (1), which is based on the mixture model (p. 1697, line 14), is constructed from

$$l_t := \log[(1-\pi)\cdot N(\theta_1 + \alpha X_{t-1}, \sigma^2) + \pi \cdot N(\theta_2 + \alpha X_{t-1}, \sigma^2)]. \tag{2}$$

To isolate the source of inconsistency in the QMLE from (2), we set the variance to 1 and let $\theta_1 = \mu$ and $\theta_2 - \theta_1 = \gamma$. The conditional density functions that enter (2) are

$$\begin{aligned} N(\mu + \alpha X_{t-1}, 1) &= f(X_t|X_{t-1}; \theta^1) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(X_t - \alpha X_{t-1} - \mu)^2\right], \\ N(\mu + \gamma + \alpha X_{t-1}, 1) &= f(X_t|X_{t-1}; \theta^2) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(X_t - \alpha X_{t-1} - \mu - \gamma)^2\right]. \end{aligned}$$

A necessary condition for consistency is that $n^{-1}\mathbb{E}[\sum_{t=1}^n l_t(\pi, \alpha, \mu, \gamma)]$ be maximized at the population parameter values. Because the process is stationary,

$$\frac{1}{n}\sum_{t=1}^n \mathbb{E}[l_t(\pi, \alpha, \mu, \gamma)] = \mathbb{E}[l_t(\pi, \alpha, \mu, \gamma)] := M(\pi, \alpha, \mu, \gamma).$$

We then have

$$\begin{aligned} M(\pi, \alpha, \mu, \gamma) &= \mathbb{E}\log[\pi\lambda(X_t, X_{t-1}) + (1-\pi)] \\ &\quad + \mathbb{E}\log f(X_t|X_{t-1}; \theta^1), \end{aligned}$$

where

$$\begin{aligned} \lambda(X_t, X_{t-1}, \theta^1, \theta^2) &= \frac{f(X_t|X_{t-1}; \theta^2)}{f(X_t|X_{t-1}; \theta^1)} \\ &= \exp\left[\gamma(X_t - \alpha X_{t-1} - \mu) - \frac{\gamma^2}{2}\right]. \end{aligned}$$

The key to establishing inconsistency is to calculate the first derivative of $M(\pi, \alpha, \mu, \gamma)$ with respect to the autoregressive coefficient α evaluated at the population values of the parameters. As we show in Carter and Steigerwald (2012), the partial derivative of the M function with respect to α is

$$\frac{\partial}{\partial\alpha} M(\pi, \alpha, \mu, \gamma) = \gamma^2 C_{\pi,\gamma}(\pi - p_{12})\left(\frac{\pi(1-\pi)}{\pi - \alpha(\pi - p_{12})}\right),$$

where $p_{12} = \mathbb{P}(S_t = 2|S_{t-1} = 1)$ and $C_{\pi,\gamma}$ is a positive constant that depends on π and γ. Therefore, if the Markov regime process includes a dependence between subsequent time points, the gradient along α is not equal to 0 at the population parameter values and the expected value of the quasi-log-likelihood is maximized away from the population parameter values.

Of course, this derivative vanishes under the null hypothesis where $\pi = 0$, $\pi = 1$, or $\gamma = 0$, as under the null hypothesis there is effectively only one regime. This derivative also vanishes for $\pi = p_{12}$ because, if $\pi = p_{12}$, then $\mathbb{P}(S_t = j) = \mathbb{P}(S_t = j|S_{t-1})$ and the Markov regime process reduces to independent draws from the stationary distribution. In this case, the mixture model (2) forms the population log-likelihood rather than the quasi-log-likelihood.

To understand the source of inconsistency, recall the classic structure that Wald (1949) proposed to demonstrate consistency of the MLE. The structure was adapted by Levine (1983) to demonstrate a general property of consistency for a QMLE, where the quasi-log-likelihood is constructed from conditional density functions. In applying the logic of Levine to Markov regime-switching processes, a key requirement is that the conditional state probabilities be independent of the regressors that enter the state-specific conditional densities. For the autoregressive process analyzed in Cho and White, the conditional state probabilities depend upon regressors that enter the state-specific conditional densities and a QMLE is inconsistent. Lack of consistency of a QMLE holds generally for autoregressive processes, as lagged values of the dependent variable, which are regressors in the conditional densities, contain information about lagged values of the state variable, which in turn contain information about the current value of the state variable.

The inconsistency of a QMLE for Markov regime-switching processes with autoregressive components extends to processes with moving-average components. Inconsistency of a QMLE, however, does not necessarily imply that a test based on the quasi-likelihood ratio (QLR) is inconsistent. Consistency of a QLR test only requires that M attain a maximum at some point outside the null hypothesis space, but not necessarily at the population parameter values. For the autoregressive process analyzed above, the gradient of the M function is zero in every coordinate except α, which indicates that M may be maximized away from the null hypothesis and that the class of models for which the QLR test is consistent is larger than the class of models for which the QMLE is consistent. A definitive treatment of consistency could be based on demonstration that, under the alternative hypothesis, the value of the likelihood is bounded over the null parameter space and that there is always a point in the alternative space for which the value of the likelihood exceeds the bound. Cho and White (2011) demonstrated this for the Gaussian AR(1) model. Even for models in which the QLR test is shown to be consistent, the power is almost certainly affected by the inconsistency of the QMLE under the alternative.

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