

SUPPLEMENT TO “TESTING FOR REGIME SWITCHING:
A COMMENT”

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We present proof of the inconsistency of the QMLE defined by Cho and White (2007). Inconsistency arises from the fact that the quasi-log-likelihood does not attain a maximum at the population parameter values.

S1. PROOF OF INCONSISTENCY OF A QMLE

IN THIS SUPPLEMENTAL section we prove that, for an autoregressive process, the gradient of the quasi-log-likelihood does not equal zero when evaluated at the population parameter values. Recall the mixture model for an autoregressive process

$$(S1) \quad (1 - \pi) \cdot N(\theta_1 + \alpha X_{t-1}, 1) + \pi N(\theta_2 + \alpha X_{t-1}, 1),$$

on which the quasi-likelihood is based. For $\theta_1 = \mu$ and $\theta_2 = \mu + \gamma$, we have

$$\frac{1}{n} \sum_{t=1}^n \mathbb{E}[l_t(\pi, \alpha, \mu, \gamma)] = \mathbb{E}[l_t(\pi, \alpha, \mu, \gamma)] := M(\pi, \alpha, \mu, \gamma),$$

where

$$M(\pi, \alpha, \mu, \gamma) = \mathbb{E} \log \left[\pi \exp \left[\gamma(X_t - \alpha X_{t-1} - \mu) - \frac{\gamma^2}{2} \right] + (1 - \pi) \right] - \frac{1}{2} [\log 2\pi + \mathbb{E}(X_t - \alpha X_{t-1} - \mu)^2].$$

Remaining variable definitions are given in the main paper.

The key is to calculate the first derivative of $M(\pi, \alpha, \mu, \gamma)$ with respect to the autoregressive coefficient α evaluated at the population values of the parameters $(\pi^*, \alpha^*, \mu^*, \gamma^*)$. To begin, let $Z_t = X_t - \alpha^* X_{t-1} - \mu^*$. The derivative for α is

$$(S2) \quad \left. \frac{\partial}{\partial \alpha} M(\pi, \alpha, \mu, \gamma) \right|_{(\pi^*, \alpha^*, \mu^*, \gamma^*)} = \mathbb{E} \left(\frac{-\pi^* \gamma^* X_{t-1} \exp \left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2} \right]}{\pi^* \exp \left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2} \right] + (1 - \pi^*)} \right) + \mathbb{E}(X_{t-1} Z_t).$$

To calculate these expectations, we must account for the correlation between Z_t and X_{t-1} . The distribution of Z_t conditional on X_{t-1} depends on this previous observation through S_{t-1} . The conditional density is

$$\begin{aligned} f(z|X_{t-1}) &= \phi(z - \gamma^*)\mathbb{P}(S_t = 2|X_{t-1}) + \phi(z)\mathbb{P}(S_t = 1|X_{t-1}) \\ &= \left[\zeta_t \exp\left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t) \right] \phi(z), \end{aligned}$$

where $\phi(z)$ is the density of a standard Gaussian random variable and $\zeta_t := \mathbb{P}(S_t = 2|\sigma(X^{t-1}))$.

The first expectation in (S2) is

$$\begin{aligned} \text{(S3)} \quad & \mathbb{E} \left(\frac{-\pi^* \gamma^* X_{t-1} \exp\left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2}\right]}{\pi^* \exp\left[\gamma^* Z_t - \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*)} \right) \\ &= -\pi^* \gamma^* \mathbb{E} \left[X_{t-1} \int \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] \right. \\ & \quad \left. \times \left(\frac{\zeta_t \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t)}{\pi^* \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*)} \right) \phi(z) dz \right]. \end{aligned}$$

Because $\exp[\gamma^* z - \frac{(\gamma^*)^2}{2}] \phi(z) = \phi(z - \gamma^*)$,

$$\begin{aligned} & \int \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] \left(\frac{\zeta_t \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t)}{\pi^* \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*)} \right) \phi(z) dz \\ &= \int \left(\frac{\zeta_t \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t)}{\pi^* \exp\left[\gamma^* z - \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*)} \right) \phi(z - \gamma^*) dz \\ &= \int \left(\frac{\zeta_t \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t)}{\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*)} \right) \phi(v) dv, \end{aligned}$$

and it follows that this integral is

$$\begin{aligned}
& \int \left(\frac{\zeta_t \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t)}{\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*)} \right) \phi(v) dv \\
&= \frac{\zeta_t}{\pi^*} + \int \left[\left(\zeta_t \pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \zeta_t) \pi^* \right. \right. \\
&\quad \left. \left. - \left(\zeta_t \pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + \zeta_t (1 - \pi^*) \right) \right) \right. \\
&\quad \left. / \left(\pi^* \left[\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*) \right] \right) \right] \phi(v) dv \\
&= \frac{\zeta_t}{\pi^*} + \frac{\pi^* - \zeta_t}{\pi^*} \int \left[\pi^* \exp\left[\gamma^* v + \frac{(\gamma^*)^2}{2}\right] + (1 - \pi^*) \right]^{-1} \phi(v) dv.
\end{aligned}$$

Therefore, substituting this expression back into (S3),

$$\begin{aligned}
& \frac{\partial}{\partial \alpha} M(\pi, \alpha, \mu, \gamma) \Big|_{(\pi^*, \alpha^*, \mu^*, \gamma^*)} \\
&= -\gamma^* \mathbb{E}(X_{t-1} \zeta_t) + \gamma^* \mathbb{E}[X_{t-1} (\zeta_t - \pi^*)] C_{\pi^*, \gamma^*} + \mathbb{E}(X_{t-1} Z_t),
\end{aligned}$$

where C_{π^*, γ^*} is the expectation

$$C_{\pi^*, \gamma^*} := \mathbb{E} \left[\pi^* \exp \left[\gamma^* \xi + \frac{(\gamma^*)^2}{2} \right] + (1 - \pi^*) \right]^{-1}$$

for ξ a standard Gaussian random variable. Clearly, C_{π^*, γ^*} does not depend on X_{t-1} or S_t and is a bounded positive quantity: $0 < C_{\pi^*, \gamma^*} \leq (1 - \pi^*)^{-1}$.

Furthermore, we can use that $\mathbb{E}(Z_t | X_{t-1}) = \gamma^* \mathbb{P}(S_t = 2 | X_{t-1}) = \gamma^* \zeta_t$ to simplify the expression

$$\frac{\partial}{\partial \alpha} M(\pi, \alpha, \mu, \gamma) \Big|_{(\pi^*, \alpha^*, \mu^*, \gamma^*)} = \gamma^* C_{\pi^*, \gamma^*} \mathbb{E}[X_{t-1} (\zeta_t - \pi^*)].$$

The last factor in this expression is

$$\mathbb{E}[X_{t-1} (\zeta_t - \pi^*)] = \mathbb{E}(X_{t-1} \zeta_t) - \mathbb{E}X_{t-1} \mathbb{E}S_t = \text{Cov}(X_{t-1}, S_t).$$

The last equality follows from $\mathbb{E}(X_{t-1} S_t) = \mathbb{E}(X_{t-1} \mathbb{E}(S_t | X_{t-1}))$ and $\mathbb{E}(S_t | X_{t-1}) = \zeta_t$.

To obtain an expression for the covariance of X_{t-1} and S_t , we use the following lemma.

LEMMA A: *Under model (S1)*

$$\text{Cov}(X_{t-1}, S_t) = \gamma(1 - \pi)(\pi - p_{12}) \left(\frac{\pi}{\pi - \alpha(\pi - p_{12})} \right).$$

Therefore, the expression for the partial derivative of the M function with respect to α becomes

$$\frac{\partial}{\partial \alpha} M(\pi, \alpha, \mu, \gamma) = \gamma^2 C_{\pi, \gamma}(\pi - p_{12}) \left(\frac{\pi(1 - \pi)}{\pi - \alpha(\pi - p_{12})} \right).$$

PROOF OF LEMMA A: We first use the recursive expression

$$X_t = \mu + \alpha X_{t-1} + \gamma S_t + \xi_t = \sum_{k=0}^{\infty} \alpha^k (\mu + \gamma S_{t-k} + \xi_{t-k}),$$

where $\xi_t \sim \text{i.i.d. } N(0, 1)$. This implies that the covariance is

$$\text{Cov}(X_{t-1}, S_t) = \gamma \sum_{k=0}^{\infty} \alpha^k \text{Cov}(S_{t-1-k}, S_t).$$

The covariance of the binary state variables is

$$\text{Cov}(S_{t-k}, S_t) = \mathbb{P}(S_t = 2 | S_{t-k} = 2) \pi - \pi^2,$$

where $\pi = \mathbb{P}(S_t = 2) = \mathbb{P}(S_{t-k} = 2)$ is the stationary probability in the Markov chain. It can be shown that the conditional probability is

$$\mathbb{P}(S_t = 2 | S_{t-k} = 2) = \pi + (1 - \pi) \left(\frac{\pi - p_{12}}{\pi} \right)^k.$$

Thus

$$\text{Cov}(S_{t-k}, S_t) = \pi(1 - \pi) \left(\frac{\pi - p_{12}}{\pi} \right)^k,$$

and

$$\begin{aligned} \text{Cov}(X_{t-1}, S_t) &= \gamma \pi(1 - \pi) \left(\frac{\pi - p_{12}}{\pi} \right) \sum_{k=0}^{\infty} \alpha^k \left(\frac{\pi - p_{12}}{\pi} \right)^k \\ &= \gamma(1 - \pi)(\pi - p_{12}) \left(\frac{\pi}{\pi - \alpha(\pi - p_{12})} \right). \end{aligned} \quad \text{Q.E.D.}$$

REFERENCE

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